JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 34, 239–266 (1971)

Necessary and Sufficient Conditions for Optimality for Singular Control Problems: A Limit Approach

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Necessary and sufficient conditions for optimality for singular control problems are presented for the case where the extremal path is totally singular. The singular second variation is converted into a nonsingular one by addition of a quadratic functional of the control; a parameter $1/\epsilon$ multiplies this added functional. By allowing ϵ to approach infinity the optimality conditions are deduced for the singular problem from the limiting optimality conditions of the synthesized nonsingular second variation. The resulting conditions are Jacobson's sufficient conditions in slightly modified form. In a companion paper necessity of Jacobson's conditions for a class of singular problems is demonstrated by exploiting the Kelley transformation technique which converts the singular second variation into a nonsingular one in a reduced dimensional state space.

I. PRELIMINARIES

1. *Introduction*

In Ref. [1] a new necessary condition of optimality for singular control problems is developed and is shown to be nonequivalent to the well-known generalized Legendre–Clebsch (or Kelley) condition. In Ref. [2] sufficient conditions for nonnegativity of the singular second variation are presented; in strengthened form these are sufficient for a weak minimum. The sufficiency

* Partially supported by ONR Contract N00014-67-A-0298-0006 and NASA Grant NGR 22-007-068.

† Supported by NASA Contract NAS 12-656. Presently with the Charles Stark Draper Lab., M.I.T.

conditions, which are in the form of linear algebraic equalities and inequalities, also yield insight into nonsingular control problems and into the behavior of the important matrix Riccati differential equation [3]. The relationship of the above new conditions of optimality to existing necessary conditions is discussed in Refs. [1, 2]; thus, we refer the reader to these papers for general references to research in singular control (variational) problems.

In this paper we show that, with slight modification, the sufficiency conditions presented in Ref. [2] are also necessary for nonnegativity of the singular second variation. Moreover, we show that these conditions are indeed necessary and sufficient for a weak minimum for a class of singular control problems. In certain cases a strong minimum is implied. We prove these results by a limit argument. The singular second variation is converted into a nonsingular one by the addition of a term $1/2\epsilon \int u^T u \, dt$, $\epsilon > 0$. By allowing ϵ to approach infinity we deduce the optimality conditions for the singular problem from the limiting optimality conditions of the synthesized nonsingular second variation. This limit approach has been used previously [4] as a computational technique for solving singular optimal control problems.

In a companion paper [5] we prove necessity of the conditions given in Ref. [2] by exploiting the Kelley transformation technique [6] which converts the singular second variation into a nonsingular second variation in a state space of reduced dimension. In the reduced state space the equivalence is established of the sufficiency conditions [2] and the existence of the solution of the Riccati differential equation (Jacobi or Conjugate Point Condition) associated with this nonsingular second variation. This proves the necessity of the conditions given in Ref. [2]. Disadvantages of the transformation technique are in its algebraic complexity and in the need for the coefficients of the second variation (which depend upon time) to be many times differentiable. Moreover, if the problem is singular of order p (i.e., $(d^{2p}/dt^{2p}) H_u$ contains the control u, where H is the variational Hamiltonian) the transformation technique must be applied repeatedly, p times, before a nonsingular problem is obtained.[1] Nevertheless, the transformation technique does give another viewpoint and ties together the conditions given in Refs. [1, 2] and the pioneering work of Kelley et al. [6] and Robbins [7]. Independently Goh[8] and McDanell and Powers [9], using Goh's transformation technique [10], have arrived at similar results.

The limit approach presented in this paper has the following advantageous features:

(i) A direct proof of necessary and sufficient conditions for optimality is obtained without the need to transform the problem to a reduced state space;

[1] A nonsingular problem is unattainable if u does not appear in a time derivative of H_u.

(ii) The concept of "order of the singular arc p" is not required in the proof;

(iii) Differentiability requirements are not as severe as those demanded in Ref. [5].

2. *Problem Formulation*

We shall consider a class of control problems where the dynamic system is described by the ordinary differential equations

$$\dot{x} = f(x, u, t), \qquad x(t_0) = x_0 \tag{1}$$

where

$$f(x, u, t) \triangleq f_1(x, t) + f_u(x, t)\, u. \tag{2}$$

The performance of the system is measured by the cost functional

$$V[u(\cdot)] = \int_{t_0}^{t_f} \{L_1(x, t) + u^T L_2(x, t)\}\, dt + F(x(t_f)), \tag{3}$$

and the terminal states must satisfy

$$\psi(x(t_f)) = 0. \tag{4}$$

The control function $u(\cdot)$ is required to satisfy the constraint

$$u(\cdot) \in U \tag{5}$$

where the set U is defined by

$$U \triangleq U_1 \cap U_2, \tag{6}$$

where

$$\begin{aligned} U_1 \triangleq \{u(\cdot) : u_{\min} \leqslant u_i(t) \leqslant u_{\max}, \ t \in [t_0, t_f], \ i = 1, \ldots, m, \\ \text{where } u_{\min} > -\infty, \ u_{\max} < +\infty\} \end{aligned} \tag{7}$$

and

$$U_2 \triangleq \{u(\cdot) : u(\cdot) \text{ is piecewise continuous in } [t_0, t_f]\}. \tag{8}$$

In the above formulation x is an n-dimensional state vector and u is an m-dimensional control vector. The function f_1 is an n-dimensional vector function of x at time t, f_u is an $n \times m$ matrix function of x at time t and L_2 is an m-dimensional vector function of x at time t; the functions L_1 and F are scalar. The terminal equality constraint function ψ is an s-dimensional column vector function of $x(t_f)$.

In the sequel we shall use the following assumptions.

ASSUMPTION 1. The initial and final times t_0, t_f are given explicitly, $-\infty < t_0 < t_f < \infty$.

ASSUMPTION 2. The functions f, L_1, L_2 are three times continuously differentiable in each argument, and the functions F and ψ are three times continuously differentiable in $x(t_f)$.

With the above formulation and assumptions in mind, we now state the optimal control problem: Determine the control function $u(\cdot)$ which satisfies (1), (4), and (5) and which minimizes $V[u(\cdot)]$.

3. *Totally Singular Problems*

Along an optimal trajectory it is well known that the following necessary conditions (Pontryagin's Principle) hold:

$$-\dot{\lambda} = H_x(\bar{x}, \bar{u}, \lambda, t), \qquad \lambda(t_f) = \lambda_0 F_x(\bar{x}(t_f)) + \psi_x{}^T \nu \tag{9}$$

where

$$\bar{u} = \underset{u_{\min} \leqslant u_i \leqslant u_{\max}}{\arg\min} \quad H(\bar{x}, u, \lambda, t) \tag{10}$$

and

$$H(x, u, \lambda, t) \triangleq \lambda_0 \{L_1(x, t) + u^T L_2(x, t)\} + \lambda^T f(x, u, t). \tag{11}$$

Here $\bar{u}(\cdot)$, $\bar{x}(\cdot)$ denote the candidate control and state functions, $\lambda(\cdot)$ is an n-vector of Lagrange multiplier functions of time, and λ_0 is a constant $\geqslant 0$. Associated with the terminal constraint ψ is an s-vector of constant Lagrange multipliers ν.

ASSUMPTION 3. We shall assume that the control problem is normal so that λ_0 can be set equal to unity.

DEFINITION 1. A totally singular arc is one along which

$$H_{u_i}(\bar{x}, \lambda, t) = 0; \qquad t \in [t_0, t_f], \quad i = 1,..., m. \tag{12}$$

We now make

ASSUMPTION 4. $\bar{u}(\cdot)$, the candidate for a minimizing solution, is continuous in t, totally singular, and

$$u_{\min} < \bar{u}_i(t) < u_{\max} \, \forall t \in [t_0, t_f], \qquad i = 1,..., m.$$

In subsequent sections we develop necessary and sufficient conditions for this totally singular control function to be a minimizing solution (relative minimum).

II. Conditions for a Relative Minimum: Unconstrained Terminal State

1. *The Second Variation* ($\delta^2 V$) *and Associated Theorems*

Before proceeding with the second variation we make the following definitions:

Definition 2. $V[u(\cdot)]$ has a weak relative minimum at $\bar{u}(\cdot)$ if

$$V[u(\cdot)] - V[\bar{u}(\cdot)] \geqslant 0$$

$$\forall u(\cdot) \in U \ni \sup_{i=1,\ldots,n} \sup_{t_0 \leqslant t \leqslant t_f} | x_i - \bar{x}_i | + \sup_{j=1,\ldots,m} \sup_{t_0 \leqslant t \leqslant t_f} | u_i - \bar{u}_i | < w.$$

Note. Since $x(t_0) - \bar{x}(t_0) = 0$ the above restriction on $u(\cdot)$ is equivalent to

$$\| u(\cdot) - \bar{u}(\cdot) \| < w_1 ,$$

where

$$\| u(\cdot) - \bar{u}(\cdot) \| \triangleq \sup_{j=1,\ldots,m} \sup_{t_0 \leqslant t \leqslant t_f} | u_i - \bar{u}_i |$$

and where $w_1 (< w)$ is sufficiently small.

Definition 3. $V[u(\cdot)]$ has a strong relative minimum at $\bar{u}(\cdot)$ if

$$V[u(\cdot)] - V[\bar{u}(\cdot)] \geqslant 0$$

$$\forall u(\cdot) \in U \ni \sup_{i=1,\ldots,n} \sup_{t_0 \leqslant t \leqslant t_f} | x_i - \bar{x}_i | < w.$$

In the sequel we shall denote $u(\cdot) - \bar{u}(\cdot)$ by $\delta u(\cdot)$.

In the absence of terminal constraints (4) an expression for the second variation (for $\delta u(t)$ sufficiently small, $t \in [t_0 , t_f]$) is

$$\delta^2 V[\delta u(\cdot)] = \int_{t_0}^{t_f} \{\tfrac{1}{2}\, \delta x^T H_{xx} \delta x + \delta u^T H_{ux} \delta x\}\, dt + \tfrac{1}{2}\, \delta x^T F_{xx} \delta x \Big|_{t_f} , \tag{13}$$

subject to the linearized differential equation

$$\delta \dot{x} = f_x \delta x + f_u \delta u, \qquad \delta x(t_0) = 0. \tag{14}$$

In (13), (14) partial derivatives are evaluated along $\bar{u}$, $\bar{x}$, λ.

We are led to the study of the second variation because of the following theorem (slightly altered form of Theorem 1, Gelfand and Fomin, Ref. [11, p. 99]):

THEOREM 1. *A necessary condition for* $V[u(\cdot)]$ *to have a weak relative minimum at* $\bar{u}(\cdot)$ *is that* $\delta^2 V[\delta u(\cdot)] \geqslant 0 \; \forall \delta u(\cdot)$ *sufficiently small to justify the expansions* (13), (14) *and such that* $\bar{u}(\cdot) + \delta u(\cdot) \in U$.

The condition $\delta^2 V[\delta u(\cdot)] \geqslant 0$ for all admissible $\delta u(\cdot)$ is necessary but not sufficient for the functional $V[u(\cdot)]$ to have a weak minimum at $\bar{u}(\cdot)$. A sufficient condition is provided by the following theorem.

THEOREM 2 (Gelfand and Fomin, Ref. [11, p. 100]). *A sufficient condition for* $V[u(\cdot)]$ *to have a weak relative minimum at* $\bar{u}(\cdot)$ *is that* $\delta^2 V[\delta u(\cdot)]$ *be strongly positive.*[2]

Unfortunately, it turns out that the singular second variation cannot be strongly positive [12]. Here we prove this using arguments similar to those suggested by Johansen [13]. For simplicity of presentation let us consider the case where x and u are scalars; the arguments generalize to the vector case. Suppose that we set $\delta u(t) = b \sin qt \; \forall t \in [t_0, t_f]$ and choose $b \ni \bar{u}(\cdot) + \delta u(\cdot) \in U$. The solution of (14) when driven by this $\delta u(\cdot)$ is

$$\delta x(t) = b \int_{t_0}^{t} \phi(t, \tau) f_u(\tau) \sin q\tau \, d\tau, \tag{15}$$

where

$$\frac{\partial}{\partial t} \phi(t, \tau) = f_x(t) \phi(t, \tau), \qquad \phi(\tau, \tau) = I. \tag{16}$$

Integrating (15) by parts yields

$$\begin{aligned} \delta x(t) = b \Big\{ \int_{t_0}^{t} \Big[\frac{\partial \phi}{\partial \tau}(t, \tau) f_u(\tau) + \phi(t, \tau) \frac{\partial f_u}{\partial \tau}(\tau) \Big] \frac{1}{q} \cos q\tau \, d\tau \\ - f_u(t) \frac{1}{q} \cos qt + \phi(t, t_0) f_u(t_0) \frac{1}{q} \cos qt_0 \Big\}. \end{aligned} \tag{17}$$

From (13),

$$\delta^2 V[\delta u(\cdot)] \leqslant \int_{t_0}^{t_f} \{ \tfrac{1}{2} \mid H_{xx} \mid \delta x^2 + \mid \delta u \mid \mid H_{ux} \mid \mid \delta x \mid \} \, dt + \tfrac{1}{2} \mid F_{xx} \mid \delta x^2 \Big|_{t_f}. \tag{18}$$

[2] That is, $\delta^2 V[\delta u(\cdot)] \geqslant k \| \delta u(\cdot) \|^2 \; \forall$ admissible $\delta u(\cdot)$, where $k > 0$ is a constant.

From (17),

$$
\begin{aligned}
|\delta x(t)| \leqslant \frac{b}{q} \Big\{ \int_{t_0}^{t} \Big| \Big[\frac{\partial \phi}{\partial \tau}(t,\tau) f_u(\tau) + \phi(t,\tau) \frac{\partial f_u}{\partial \tau}(\tau) \Big] \Big| \, |\cos q\tau| \, d\tau \\
+ |f_u(t)| \, |\cos qt| + |\phi(t,t_0)| \, |f_u(t_0)| \, |\cos qt_0| \Big\}
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
\leqslant \frac{b}{q} \Big\{ \int_{t_0}^{t} \Big| \Big[\frac{\partial \phi}{\partial \tau}(t,\tau) f_u(\tau) + \phi(t,\tau) \frac{\partial f_u}{\partial \tau}(\tau) \Big] \Big| \, d\tau \\
+ |f_u(t)| + |\phi(t,t_0)| \, |f_u(t_0)| \Big\}
\end{aligned} \tag{20}
$$

$$
\triangleq \frac{b}{q} \cdot \beta(t). \tag{21}
$$

Thus

$$
\begin{aligned}
\delta^2 V[\delta u(\cdot)] \leqslant \int_{t_0}^{t_f} \Big\{ \frac{1}{2} |H_{xx}| \Big(\frac{b}{q}\Big)^2 \beta^2 + \frac{b^2}{q} \beta \, |H_{ux}| \Big\} \, dt \\
+ \frac{1}{2} |F_{xx}| \Big(\frac{b}{q}\Big)^2 \beta^2 \Big|_{t_f}
\end{aligned} \tag{22}
$$

$$
\triangleq b^2 \Big\{ \frac{d_1}{q^2} + \frac{d_2}{q} \Big\}. \tag{23}
$$

Now if $\delta^2 V[\delta u(\cdot)]$ is strongly positive then $\exists k$ (constant) > 0 such that

$$
b^2 \Big\{ \frac{d_1}{q^2} + \frac{d_2}{q} \Big\} \geqslant k \, \| \delta u(\cdot) \|^2, \qquad \forall q, \, 0 < q < \infty. \tag{24}
$$

Clearly (24) cannot hold for k a constant (> 0), since q can be made arbitrarily large (q is directly proportional to the frequency of the sine input). Thus $\delta^2 V[\delta u(\cdot)]$ cannot be strongly positive.

In view of the above failure of Theorem 2 for singular problems we offer the following more useful sufficiency theorem.

THEOREM 3. *A sufficient condition for $V[u(\cdot)]$ to have a minimum at $\bar{u}(\cdot)$ is that $\delta^2 V[\delta u(\cdot)] \geqslant 0$ for all admissible $\delta u(\cdot)$ and that*

$$
\delta^2 V[\delta u(\cdot)] \geqslant | \alpha[\delta u(\cdot)] | \, \| \delta u(\cdot) \|^2 \tag{25}
$$

for $\| \delta u(\cdot) \|$ sufficiently small, where

$$
\alpha[\delta u(\cdot)] \triangleq \frac{V[\bar{u}(\cdot) + \delta u(\cdot)] - V[\bar{u}(\cdot)] - \delta^2 V[\delta u(\cdot)]}{\| \delta u(\cdot) \|^2}. \tag{26}
$$

Proof. Let

$$\Delta V[\delta u(\cdot)] \triangleq V[\bar{u}(\cdot) + \delta u(\cdot)] - V[\bar{u}(\cdot)];$$

then, by Assumption 2,

$$\Delta V[\delta u(\cdot)] = \delta^2 V[\delta u(\cdot)] + \alpha[\delta u(\cdot)] \, \| \delta u(\cdot)\|^2, \tag{27}$$

where $\alpha[\delta u(\cdot)] \to 0$ as $\| \delta u(\cdot)\| \to 0$. From (27) and (25),

$$\Delta V[\delta u(\cdot)] - \alpha[\delta u(\cdot)] \, \| \delta u(\cdot)\|^2 = \delta^2 V[\delta u(\cdot)] \geqslant | \alpha[\delta u(\cdot)]| \, \| \delta u(\cdot)\|^2$$

for $\| \delta u(\cdot)\|$ sufficiently small. Thus,

$$\Delta V[\delta u(\cdot)] \geqslant \| \delta u(\cdot)\|^2 \{| \alpha[\delta u(\cdot)]| + \alpha[\delta u(\cdot)]\} \tag{28}$$

so that

$$\Delta V[\delta u(\cdot)] \geqslant 0 \qquad \forall \delta u(\cdot) \ni \| \delta u(\cdot)\| \quad \text{sufficiently small.} \tag{29}$$

However, (29) is just the definition that $V[u(\cdot)]$ has a weak relative minimum at $\bar{u}(\cdot)$, so that the theorem is proved.

Note. If, as in the nonsingular control problem, the conditions of Theorem 2 are satisfied, then clearly so are those of Theorem 3. However, the important point is that Theorem 3 can be satisfied without $\delta^2 V[\delta u(\cdot)]$ being strongly positive.

2. *Principal Results*

Before stating our main theorems we make the following assumption and definition.

Assumption 5. The linearized system $\delta \dot{x} = f_x \delta x + f_u \delta u$ is controllable from time t_0 to time τ, $\forall \tau \in (t_0, t_f]$; that is,

$$\int_{t_0}^{\tau} \phi(\tau, \sigma) f_u(\sigma) f_u^T(\sigma) \phi^T(\tau, \sigma) \, d\sigma > 0 \quad \forall \tau \in (t_0, t_f], \tag{30}$$

where

$$\frac{\partial}{\partial \tau} \phi(\tau, \sigma) = f_x(\tau) \phi(\tau, \sigma), \qquad \phi(\sigma, \sigma) = I. \tag{31}$$

Definition 4. A real symmetric $n \times n$ matrix function of time $\Theta(t)$ is said to be monotone increasing in t if the scalar $J(x, t) \triangleq x^T \Theta(t) x$ is monotone increasing in t for all constant n-vectors x.

THEOREM A1.

(i) *Necessary Condition. Under Assumption 1–5 a necessary condition for* $\delta^2 V[\delta u(\cdot)] \geqslant 0$ *for all admissible* $\delta u(\cdot)$ *is that* $\exists \forall t \in (t_0, t_f]$ *a real symmetric matrix function of time* $\hat{P}(\cdot)$ *which is monotone increasing in t such that*

$$H_{ux} + f_u^T P = 0 \qquad \forall t \in (t_0, t_f], \tag{32}$$

$$F_{xx}(t_f) - P(t_f) = -\hat{P}(t_f) \geqslant 0, \tag{33}$$

where

$$P = Q + \Phi^T \hat{P} \Phi, \tag{34}$$

$$\dot{\Phi} = -\Phi f_x, \qquad \Phi(t_f) = I, \tag{35}$$

and

$$-\dot{Q} = H_{xx} + f_x^T Q + Q f_x, \qquad Q(t_f) = F_{xx}(t_f). \tag{36}$$

(ii) *Sufficient Condition. In addition to the above-stated condition* $\hat{P}(\cdot)$ *exists* $\forall t \in [t_0, t_f]$ *(strengthened existence condition.)*

Note. The gap between the necessary and the sufficient condition is minimal.

COROLLARY A1.1. *If* L_1 *and F are quadratic functions of x, if* f_1 *and* L_2 *are linear in x, and if* f_u *is independent of x, then condition* (i) *of Theorem* A1 *is necessary and together with the strengthened existence condition* (ii) *of Theorem* A1 *is sufficient for* $V[u(\cdot)]$ *to have a strong minimum at* $\bar{u}(\cdot)$.

COROLLARY A1.2. [2]. *A sufficient condition for* $\delta^2 V[\delta u(\cdot)] \geqslant 0$ *for all admissible* $\delta u(\cdot)$ *is that* $\exists \forall t \in [t_0, t_f]$ *a real symmetric matrix function of time* $P(\cdot)$ *which is continuously differentiable, such that*

$$H_{ux} + f_u^T P = 0 \qquad \forall t \in [t_0, t_f], \tag{37}$$

$$\dot{P} + P f_x + f_x^T P + H_{xx} = M(t) \geqslant 0 \qquad \forall t \in [t_0, t_f], \tag{38}$$

$$F_{xx}(t_f) - P(t_f) \geqslant 0. \tag{39}$$

Proof. Integration of (38) and the use of (34), (35), and (36) yields

$$\frac{d}{dt}\hat{P} \triangleq [\Phi^{-1}(t)]^T M(t) [\Phi^{-1}(t)] \geqslant 0 \Rightarrow \hat{P}$$

monotone increasing.

Note. The gap between this sufficient condition and the sufficient condition of Theorem A1 appears to be minimal since $\hat{P}$ is, a priori, differentiable almost everywhere (because it is monotone [14]). However, one can construct nonnegative quadratic functionals which satisfy the conditions of Theorem A1 but not the conditions of Corollary A1.2.

THEOREM A2. *Sufficient conditions for $V[u(\cdot)]$ to have a weak relative minimum at $\bar{u}(\cdot)$ are that the strengthened condition* (ii) *of Theorem* A1 *holds and that*

$$\delta^2 V[\delta u(\cdot)] \geqslant 0 \Rightarrow V[\bar{u}(\cdot) + \delta u(\cdot)] \geqslant V[\bar{u}(\cdot)].$$

See Theorem 3 for a condition which ensures this.

3. *Useful Lemmas*

We need the following Lemmas in order to prove Theorem A1.

LEMMA 1. *The second variation* (13), (14) *is expressible in the equivalent canonical form*

$$\delta^2 V[\delta u(\cdot)] = \delta^2 \hat{V}[\delta u(\cdot)] \triangleq \int_{t_0}^{t_f} \delta u^T C(t)\, \delta y\, dt \tag{40}$$

subject to

$$\frac{d}{dt}(\delta y(t)) = B(t)\, \delta u(t), \qquad \delta y(t_0) = 0, \tag{41}$$

where

$$C(t) = [H_{ux}(t) + f_u^T(t)\, Q(t)]\, \Phi^{-1}(t), \tag{42}$$

$$B(t) = \Phi(t)\, f_u(t), \tag{43}$$

$$\dot{\Phi}(t) = -\Phi(t)\, f_x(t), \qquad \Phi(t_f) = I, \tag{44}$$

and where

$$-\dot{Q}(t) = H_{xx}(t) + Q(t)\, f_x(t) + f_x^T(t)\, Q(t), \qquad Q(t_f) = F_{xx}(\bar{x}(t_f)), \tag{45}$$

$$\delta y(t) = \Phi(t)\, \delta x(t). \tag{46}$$

Proof. (i) (13), (14) $\Rightarrow$ (40)–(46). Adjoin the linearized system equation (14) to the second variation (13) using a continuously differentiable vector multiplier function of time $\delta\lambda(\cdot)$,

$$\begin{aligned} \delta^2 \hat{V}[\delta u(\cdot)] = \int_{t_0}^{t_f} \{\tfrac{1}{2}\, \delta x^T H_{xx} \delta x + \delta u^T H_{ux} \delta x + \delta\lambda^T (f_x \delta x + f_u \delta u - \delta\dot{x})\}\, dt \\ + \tfrac{1}{2}\, \delta x^T F_{xx} \delta x \Big|_{t_f} . \end{aligned} \tag{47}$$

Integrating $\delta\lambda\, \delta\dot{x}$ by parts yields

$$\delta^2\hat{V}[\delta u(\cdot)] = \int_{t_0}^{t_f} \{\tfrac{1}{2}\, \delta x^T H_{xx}\delta x + \delta u^T H_{ux}\delta x + \delta\lambda^T(f_x\delta x + f_u\delta u) + \dot{\delta\lambda}\delta x\}\, dt$$
$$+ \tfrac{1}{2}\, \delta x^T F_{xx}\delta x \Big|_{t_f} - [\delta\lambda^T\delta x]_{t_0}^{t_f}. \tag{48}$$

Now set

$$\delta\lambda(t) = \tfrac{1}{2}\, Q(t)\, \delta x(t), \tag{49}$$

where $Q(t)$ is an $n \times n$ symmetric matrix (continuously differentiable) function of time. Noting that $\delta x(t_0) \triangleq 0$, the second variation becomes

$$\delta^2\hat{V}[\delta u(\cdot)] = \int_{t_0}^{t_f} \{\tfrac{1}{2}\, \delta x^T(\dot{Q} + Qf_x + f_x{}^T Q + H_{xx})\, \delta x + \delta u^T(H_{ux} + f_u{}^T Q)\, \delta x\}\, dt$$
$$+ \tfrac{1}{2}\, \delta x^T(F_{xx} - Q)\, \delta x \Big|_{t_f} \tag{50}$$

Now we let

$$-\dot{Q} = H_{xx} + f_x{}^T Q + Qf_x\,, \qquad Q(t_f) = F_{xx}(t_f). \tag{51}$$

Clearly $Q(\cdot)$ is a well-defined continuously differentiable function of time. With this choice of $Q(\cdot)$ the second variation is

$$\delta^2\hat{V}[\delta u(\cdot)] = \int_{t_0}^{t_f} \delta u^T(H_{ux} + f_u{}^T Q)\, \delta x\, dt \tag{52}$$

subject to

$$\delta\dot{x} = f_x\delta x + f_u\delta u, \qquad \delta x(t_0) = 0. \tag{53}$$

Define

$$\delta y = \Phi\delta x, \tag{54}$$

where

$$\dot{\Phi} = -\Phi f_x\,, \qquad \Phi(t_f) = I. \tag{55}$$

Clearly $\Phi(t)$ is invertible $\forall t \in [t_0\,, t_f]$, so that the second variation becomes

$$\delta^2\hat{V}[\delta u(\cdot)] = \int_{t_0}^{t_f} \delta u^T(H_{ux} + f_u{}^T Q)\, \Phi^{-1}\delta y\, dt \tag{56}$$

subject to

$$\delta\dot{y} = \Phi f_u\delta u, \qquad \delta y(t_f) = 0, \tag{57}$$

which is the required form of the second variation.

(ii) (40)–(46) $\Rightarrow$ (13), (14). Since Φ is invertible, (54), (56), (57) yield (52), (53). Adjoint (53) to (52) with a Lagrange multiplier function $\delta\tilde{\lambda}(\cdot)$ and set

$$\delta\tilde{\lambda}(t) = \tfrac{1}{2}\tilde{Q}(t)\,\delta x(t). \tag{58}$$

The second variation becomes

$$\begin{aligned}\delta^2\tilde{V}[\delta u(\cdot)] = \int_{t_0}^{t_f} \{\tfrac{1}{2}\,\delta x^T(\dot{\tilde{Q}} + \tilde{Q}f_x + f_x{}^T\tilde{Q})\,\delta x + \delta u^T(H_{ux} + f_u{}^TQ + f_u{}^T\tilde{Q})\,\delta x\}\,dt \\ - \tfrac{1}{2}\,\delta x^T\tilde{Q}\delta x\Big|_{t_f}.\end{aligned} \tag{59}$$

Letting

$$\tilde{Q}(t) = -Q(t) \qquad \forall t \in [t_0, t_f] \tag{60}$$

yields

$$\delta^2\tilde{V}[\delta u(\cdot)] = \int_{t_0}^{t_f} \{\tfrac{1}{2}\,\delta x^T H_{xx}\delta x + \delta u^T H_{xx}\delta x\}\,dt + \tfrac{1}{2}\,\delta x^T F_{xx}\delta x\Big|_{t_f} \tag{61}$$

so that the Lemma is proved.

LEMMA 2. *Condition* (i) *of Theorem* A1 *is equivalent to the existence* $\forall t \in (t_0, t_f]$ *of a real symmetric monotone increasing matrix function of time* $\hat{P}(\cdot)$ *such that*

$$C + B^T\hat{P} = 0 \qquad \forall t \in (t_0, t_f], \tag{62}$$

$$-\hat{P}(t_f) \geqslant 0. \tag{63}$$

Proof. Let

$$P = \Phi^T\hat{P}\Phi + Q. \tag{64}$$

Substituting this into (62), (63) yields (32), (33) and vice versa. Note the coordinate independence of conditions (32), (33).

4. *A Related Nonsingular Second Variation*

Consider the nonsingular quadratic functional

$$\delta^2 V_N[\delta u(\cdot), \epsilon] = \delta^2\hat{V}[\delta u(\cdot)] + \frac{1}{2\epsilon}\int_{t_0}^{t_f} \delta u^T\delta u\,dt \tag{65}$$

$$= \int_{t_0}^{t_f} \left\{\delta u^T C\delta y + \frac{1}{2\epsilon}\,\delta u^T\delta u\right\} dt \tag{66}$$

where $\epsilon > 0$ is a scalar and

$$\delta\dot{y} = B\delta u, \qquad \delta y(t_0) = 0. \tag{67}$$

LEMMA 3. *If $\delta^2\hat{V}[\delta u(\cdot)]$ is nonnegative $\forall\delta u(\cdot)$ then so is $\delta^2 V_N[\delta u(\cdot), \epsilon]$ nonnegative. Moreover $\delta^2 V_N[\delta u(\cdot), \epsilon]$ is positive definite.*

Proof.

$$\frac{1}{2\epsilon}\int_{t_0}^{t_f} \delta u^T \delta u \, dt > 0 \qquad \text{unless} \qquad \delta u(\cdot) = 0.$$

THEOREM 4 (Gelfand and Fomin, Ref. [11, p. 123], Breakwell and Ho, Ref. [15]). *If the quadratic functional $\delta^2\hat{V}[\delta u(\cdot)]$ is nonnegative then the matrix Riccati differential equation*

$$-\dot{S}_\epsilon = -(C + B^T S_\epsilon)^T (C + B^T S_\epsilon)\,\epsilon, \qquad S_\epsilon(t_f) = 0, \tag{68}$$

associated with $\delta^2 V_N[\delta u(\cdot), \epsilon]$ has a solution which exists $\forall t \in [t_0, t_f]$.

Proof. By Lemma 3, $\delta^2 V_N[\delta u(\cdot), \epsilon]$ is positive definite if $\delta^2\hat{V}[\delta u(\cdot)]$ is nonnegative $\forall\delta u(\cdot)$; Gelfand and Fomin's theorem for positive definite functionals then applies.

Note. A special case of (66), (67) is treated in Ref. [11]; see Ref. [15] for the general case.

THEOREM 5. *If the matrix Riccati differential equation* (68) *has a solution which exists in the interval $[\tau, t_f]$, $t_0 \leqslant \tau \leqslant t_f$, then the control function that minimizes*

$$\delta^2 V_N{}^\tau[\delta y(\tau), \delta u(\cdot), \epsilon, \tau] \triangleq \int_\tau^{t_f} \left\{ \delta u^T C \delta y + \frac{1}{2\epsilon}\, \delta u^T \delta u \right\} dt \tag{69}$$

subject to

$$\delta\dot{y} = B\delta u, \qquad \delta y(\tau) \text{ given} \tag{70}$$

is

$$\delta u^0(t) = -\,\epsilon[C(t) + B^T(t)\, S_\epsilon(t)]\, \delta y(t), \qquad t \in [\tau, t_f], \tag{71}$$

and moreover,

$$\begin{aligned}\min_{\delta u(\cdot)} \delta^2 V_N{}^\tau[\delta y(\tau), \delta u(\cdot), \epsilon, \tau] &\triangleq \delta^2 V^0[\delta y(\tau), \epsilon, \tau] \\ &= \tfrac{1}{2}\, \delta y^T(\tau)\, S_\epsilon(\tau)\, \delta y(\tau).\end{aligned} \tag{72}$$

Proof. Substitute (72) into the Bellman equation

$$-\frac{\partial}{\partial t}\{\delta^2 V^0[\delta y(t), \epsilon, t]\} = \min_{\delta u}\Big[\delta u^T C \delta y + \frac{1}{2\epsilon}\,\delta u^T \delta u + \Big\{\frac{\partial}{\partial(\delta y)}\,\delta^2 V^0[\delta y(t), \epsilon, t]\Big\}^T B\delta u\Big]. \tag{73}$$

Minimizing with respect to δu yields (71) and causes the right side of (73) to be quadratic in $\delta y(t)$. Equating coefficients of quadratic terms in (73) yields equation (68) for $\dot{S}_\epsilon$. If $S_\epsilon(t)$ exists $\forall t \in [t_0, t_f]$ Bellman's equation is valid and the theorem is proved.

LEMMA 4. *$S_\epsilon(\cdot)$ is a continuous function of the parameter ϵ.*

Proof. C and B are continuous functions of time in $[t_0, t_f]$ and the right side of (68) is analytic in S_ϵ and ϵ (Coddington and Levinson, Ref. [16]).

LEMMA 5. *$\delta^2 V^0[\delta y(\tau), \epsilon, \tau] = \frac{1}{2}\,\delta y^T(\tau)\, S_\epsilon(\tau)\, \delta y(\tau)$ is a monotone decreasing function of ϵ (ϵ increasing); $\tau \in [t_0, t_f]$. Moreover, $S_\epsilon(\tau)$ is a monotone decreasing matrix function of ϵ; $\tau \in [t_0, t_f]$.*

Proof. For some arbitrary $\delta y(\tau)$, τ, and ϵ_1, we have

$$\delta^2 V^0[\delta y(\tau), \epsilon_1, \tau] = \min_{\delta u(\cdot)} \int_\tau^{t_f} \Big\{\delta u^T C \delta y + \frac{1}{2\epsilon_1}\,\delta u^T \delta u\Big\}\, dt. \tag{74}$$

Let the control function that minimizes (74) be $\delta u_1^0(\cdot)$ and its associated state path be $\delta y_1^0(\cdot)$. Now for any $\epsilon_2 \geqslant \epsilon_1$ it is clear that

$$\int_\tau^{t_f} \Big\{(\delta u_1^0)^T C \delta y_1^0 + \frac{1}{2\epsilon_2}(\delta u_1^0)^T \delta u_1^0\Big\}\, dt \leqslant \int_\tau^{t_f} \Big\{(\delta u_1^0)^T C \delta y_1^0 + \frac{1}{2\epsilon_1}(\delta u_1^0)^T \delta u_1^0\Big\}\, dt \tag{75}$$

and, by definition,

$$\delta^2 V^0[\delta y(\tau), \epsilon_2, \tau] \leqslant \int_\tau^{t_f} \Big\{(\delta u_1^0)^T C \delta y_1^0 + \frac{1}{2\epsilon_2}(\delta u_1^0)^T \delta u_1^0\Big\}\, dt. \tag{76}$$

Thus for any $\epsilon_2 \geqslant \epsilon_1$ we have

$$\delta^2 V^0[\delta y(\tau), \epsilon_2, \tau] \leqslant \delta^2 V^0[\delta y(\tau), \epsilon_1, \tau]. \tag{77}$$

Since $\delta y(\tau)$ and τ are arbitrary, the first part of the theorem is proved. That $S_{\epsilon_2}(\tau) \leqslant S_{\epsilon_1}(\tau)$, $\tau \in [t_0, t_f]$, $\epsilon_2 \geqslant \epsilon_1$, follows from (77) and Theorem 5.

LEMMA 6. *Under Assumptions 1–5, if $\delta^2 \hat{V}[\delta u(\cdot)] \geqslant 0$ for all admissible $\delta u(\cdot)$ then $S_\infty(t) \triangleq \lim_{\epsilon\to\infty} S_\epsilon(t)$ exists $\forall t \in (t_0, t_f]$ and is negative semidefinite.*

Proof. From (68) $S_\epsilon(\tau) \leqslant 0 \; \forall \tau \in [t_0, t_f]$, $\forall \epsilon$, $0 < \epsilon < \infty$. By Lemma 5 $S_\epsilon(\tau)$ is a monotone decreasing function of ϵ (ϵ increasing) so that it has a limit (possibly $-\infty$).

Given an arbitrary time τ in the interval (t_0, t_f) we can, by Assumption 5, construct a variation $\delta u_3(t)$, $t \in [t_0, \tau)$, such that

$$\int_{t_0}^{\tau} \left\{ \delta u_3{}^T C \delta y_3 + \frac{1}{2\epsilon} \delta u_3{}^T \delta u_3 \right\} dt$$

is finite and such that

$$\delta y_3(\tau) = \gamma(\tau), \qquad \gamma(\tau) \quad \text{arbitrary.} \tag{78}$$

Suppose that $S_\infty(\tau) = -\infty$. Then by Lemmas 4 and 5, $S_\epsilon(\tau)$ can be made arbitrarily large and negative by increasing ϵ. This implies that for some $\gamma(\tau)$, $\delta^2 V^0[\delta y_3(\tau), \epsilon, \tau]$ can be made arbitrarily large and negative which implies that

$$\int_{t_0}^{\tau} \left\{ \delta u_3{}^T C \delta y_3 + \frac{1}{2\epsilon} \delta u_3{}^T \delta u_3 \right\} dt + \delta^2 V^0[\delta y_3(\tau), \epsilon, \tau] < 0 \tag{79}$$

for some ϵ, $0 < \epsilon < \infty$. By Lemma 3 this contradicts the assumption that $\delta^2 \hat{V}[\delta u(\cdot)] \geqslant 0$ so that $S_\infty(\tau)$ cannot be infinite. Since τ is arbitrary, and since $S_\epsilon(t_f) = 0 \; \forall \epsilon$, we conclude that

$$S_\infty(\tau), \qquad \tau \in (t_0, t_f] \quad \text{is finite.} \tag{80}$$

Note. We have not shown that as ϵ becomes large the control $\{\bar{u}(t) + \delta u_3(t),\ t \in [t_0, \tau);\ \bar{u}(t) + \delta u^0(t),\ t \in [\tau, t_f]\} \in U$. However, we have shown that if $S_\infty(\cdot)$ is not finite in the interval $(t_0, t_f]$ then $\delta^2 V_N[\delta y(t_0), \delta u(\cdot), \epsilon, t_0]$, ϵ sufficiently large but finite, does not have a minimum at $\delta u(\cdot) = 0$. Since for the quadratic functional there is no distinction between a weak and a strong minimum it follows that $\delta^2 V_N[\delta y(t_0), \delta u(\cdot), \epsilon, t_0]$ can be made negative by a weak variation which satisfies $\bar{u}(\cdot) + \delta u(\cdot) \in U$. We conclude, then, that the Lemma is proved.

LEMMA 7. *$S_\infty(\tau)$ is a monotone increasing matrix function of τ.*

Proof. From (68)

$$S_\infty(\tau) = \lim_{\epsilon \to \infty} \left[\int_{t_0}^{\tau} (C + B^T S_\epsilon)^T (C + B^T S_\epsilon)\, \epsilon \, dt \right] \tag{81}$$

$$\begin{aligned} &= \lim_{\epsilon \to \infty} \left[\int_{t_0}^{\tau - \Delta} (C + B^T S_\epsilon)^T (C + B^T S_\epsilon)\, \epsilon \, dt \right] \\ &\quad + \lim_{\epsilon \to \infty} \left[\int_{\tau - \Delta}^{\tau} (C + B^T S_\epsilon)^T (C + B^T S_\epsilon)\, \epsilon \, dt \right]. \end{aligned} \tag{82}$$

From (82)

$$S_\infty(\tau) = S_\infty(\tau - \Delta) + \lim_{\epsilon\to\infty} \left[\int_{\tau-\Delta}^{\tau} (C + B^T S_\epsilon)^T (C + B^T S_\epsilon)\, \epsilon\, dt\right]. \tag{83}$$

The validity of the Lemma follows immediately from (83) since for all $\epsilon > 0$,

$$\left[\int_{\tau-\Delta}^{\tau} (C + B^T S_\epsilon)^T (C + B^T S_\epsilon)\, \epsilon\, dt\right] \geqslant 0 \qquad \forall\Delta \geqslant 0. \tag{84}$$

LEMMA 8. *If $S_\infty(\tau)$ exists for $\tau \in (t_0, t_f]$ then*

$$C(\tau) + B^T(\tau)\, S_\infty(\tau) = 0 \qquad \text{a.e. } in\ [t_0, t_f]. \tag{85}$$

Proof. Suppose the contrary. Then for some $\tau \in (t_0, t_f)$, by Assumption 2 and Lemma 4, $\exists\epsilon^* > 0 \ni$

$$-\int_{t_f}^{\tau} (C + B^T S_\epsilon)^T (C + B^T S_\epsilon)\, dt > 0 \qquad \forall\epsilon \geqslant \epsilon^*, \tag{86}$$

so that

$$\lim_{\epsilon\to\infty} \left[-\int_{t_f}^{\tau} (C + B^T S_\epsilon)^T (C + B^T S_\epsilon)\, \epsilon\, dt\right] = \infty, \tag{87}$$

contradicting the fact that $S_\infty(\tau)$ exists for $\tau \in (t_0, t_f]$.

5. *Proof of Theorem* A1

(i) *Necessary Condition.* If Assumptions 1–5 are satisfied and if $\delta^2\hat{V}[\delta u(\cdot)] \geqslant 0$ for all admissible $\delta u(\cdot)$, then by Lemmas 6–8 $\exists S_\infty(t)$, $t \in (t_0, t_f]$, which is a real symmetric monotone increasing matrix function of time such that

$$C + B^T S_\infty = 0 \qquad \text{a.e. in } [t_0, t_f], \tag{88}$$

$$S_\infty(t_f) = 0. \tag{89}$$

We show now that (88), (89) imply (62), (63) which by Lemma 2 yield the conditions of Theorem A1.

Define

$$\hat{P}(t_f) = S_\infty(t_f^-). \tag{90}$$

Since by Lemma 7, $S_\infty(t)$ is a monotone increasing function of t it follows that

$$-\hat{P}(t_f) \geqslant 0. \tag{91}$$

Defining

$$\hat{P}(t) = S_\infty(t), \qquad t \in (t_0, t_f), \tag{92}$$

yields

$$C + B^T\hat{P} = 0 \qquad \text{a.e. in } [t_0, t_f], \tag{93}$$

where

$$\hat{P}(t) \qquad \text{exists} \qquad \forall t \in (t_0, t_f] \tag{94}$$

and is monotone increasing in t. In order to establish necessity of (62) we need only show that (93) implies

$$C + B^T\hat{P} = 0 \qquad \forall t \in (t_0, t_f]. \tag{95}$$

Suppose that for some $t \in (t_0, t_f)$

$$C(t) + B^T(t)\,\hat{P}(t) \neq 0. \tag{96}$$

Since C and B are continuous functions of time (96) can only occur if a jump in $\hat{P}$ occurs at time t that does not lie in the null space of $B^T(t)$. Moreover, since $\hat{P}$ is monotone increasing in t it follows from (96) that

$$C(t)\,B(t) + B^T(t)\,\hat{P}(t)\,B(t) > 0. \tag{97}$$

Now since $\hat{P}$ is monotone increasing in t, and C and B are continuous functions of time, (97) must hold during the time interval $[t, t + \Delta]$ ($\Delta > 0$ and sufficiently small), contradicting (93). Thus, we are led to the conclusion that

$$C + B^T\hat{P} = 0 \qquad \forall t \in (t_0, t_f). \tag{98}$$

Equation (95) follows from (98) since $\hat{P}(t_f^-) \equiv \hat{P}(t_f)$.

(ii) *Sufficient Condition.* Suppose that (91) and (95) are satisfied and that (94) is satisfied in strengthened form (i.e., $\hat{P}(t)$ exists $\forall t \in [t_0, t_f]$). Adjoin the linearized dynamics to the second variation as follows:

$$\delta^2\hat{V}[\delta u(\cdot)] = \delta^2\tilde{V}[\delta u(\cdot)] \triangleq \int_{t_0}^{t_f} \{\delta u^T C \delta y + \delta y^T \hat{P}(B\delta u - \delta\dot{y})\}\, dt \tag{99}$$

$$= \int_{t_0}^{t_f} \delta u^T(C + B^T\hat{P})\,\delta y\, dt - \int_{t_0}^{t_f} \delta y^T \hat{P} \delta\dot{y}\, dt. \tag{100}$$

The first integral is zero since (95) holds. The remaining integral can be written in Stieltjies form as

$$-\int_{t_0}^{t_f} \delta y^T \hat{P}\, d(\delta y) \tag{101}$$

which, upon integration by parts (Ref. [14], p. 118), becomes

$$\int_{t_0}^{t_f} [\tfrac{1}{2}\, \delta y^T \, d\hat{P}(t)\, \delta y - \tfrac{1}{2}\, \delta y^T \hat{P} \delta y]_{t_0}^{t_f}. \tag{102}$$

Thus

$$\delta^2 \tilde{V}[\delta u(\cdot)] = \int_{t_0}^{t_f} \tfrac{1}{2}\, \delta y^T \, d\hat{P}(t)\, \delta y - \tfrac{1}{2}\, \delta y^T \hat{P} \delta y \Big|_{t_f} \tag{103}$$

since $\delta y(t_0) = 0$. From (91) the term evaluated at t_f is nonnegative, and the integral is nonnegative since $\hat{P}(\cdot)$ is a monotone increasing matrix function of time. Sufficiency is proved.

III. Conditions for Relative Minimum: Constrained Terminal State

1. *The Second Variation* $\delta^2 V^*$

Here we treat the class of totally singular problems where equality (4) is present. In this case the second variation (for $\delta u(t)$ sufficiently small, $t \in [t_0, t_f]$) has the form

$$\begin{aligned} \delta^2 V^*[\delta u(\cdot)] = \int_{t_0}^{t_f} \{\tfrac{1}{2}\, \delta x^T H_{xx} \delta x + \delta u^T H_{ux} \delta x\}\, dt \\ + \tfrac{1}{2}\, \delta x^T (F_{xx} + \nu^T \psi_{xx})\, \delta x \Big|_{t_f} \end{aligned} \tag{104}$$

subject to the linearized differential equation

$$\delta \dot{x} = f_x \delta x + f_u \delta u, \qquad \delta x(t_0) = 0 \tag{105}$$

and the linearized terminal constraint

$$\psi_x \delta x \Big|_{t_f} = 0. \tag{106}$$

We now introduce

Assumption 6. The $s \times n$ matrix $\psi_x(\bar{x}(t_f))$ has full rank s.

Lemma 9. *By Assumption 6, s components of $\delta x(t_f)$, referred to as $\delta x^s(t_f)$, can be solved for in terms of the remaining $n - s$ components $\delta x^{n-s}(t_f)$.*

For example,

$$\delta x^s(t_f) = - A_1^{-1} A_2 \delta x^{n-s}(t_f), \tag{108}$$

where[3]

$$\begin{array}{c} \\ s \updownarrow \end{array} \overset{\xleftrightarrow{\;n\;}}{[\underset{\xleftrightarrow{s}}{A_1} \mid \underset{(n-s)}{A_2}]} = \psi_x \tag{109}$$

so that

$$\delta x(t_f) = \left[\frac{-A_1^{-1}A_2\delta x^{n-s}(t_f)}{\delta x^{n-s}(t_f)}\right] = \left[\frac{-A_1^{-1}A_2}{I}\right] \delta x^{n-s}(t_f) \triangleq Z\delta x^{n-s}(t_f), \tag{110}$$

where Z is $n \times (n - s)$.

2. *Principal Results*

THEOREM B1.

(i) *Necessary Condition. Under Assumptions* 1–6 *a necessary condition for* $\delta^2 V^*[\delta u(\cdot)] \geqslant 0$ *for all admissible* $\delta u(\cdot)$ *is that* $\exists \forall t \in (t_0, t_f)$ *a real symmetric matrix function of time* $\hat{P}^*(\cdot)$ *which is monotone increasing in t such that*

$$H_{ux} + f_u{}^T P^* = 0 \qquad \forall t \in (t_0, t_f), \tag{111}$$

$$Z^T(F_{xx} + \nu^T\psi_{xx} - P^*(t_f^-))\, Z = -\, Z^T\hat{P}^*(t_f^-)\, Z \geqslant 0, \tag{112}$$

where

$$P^* = Q + \Phi^T\hat{P}^*\Phi, \tag{113}$$

$$\dot{\Phi} = -\,\Phi f_x\,, \qquad \Phi(t_f) = I, \tag{114}$$

$$-\dot{Q} = H_{xx} + f_x{}^T Q + Q f_x\,, \qquad Q(t_f) = F_{xx} + \nu^T\psi_{xx}\,, \tag{115}$$

and where Z is defined in (110).

(ii) *Sufficient Condition. In addition to the above-stated condition* $\hat{P}^*(t)$ *exists* $\forall t \in [t_0, t_f]$ *and*

$$Z^T(F_{xx} + \nu^T\psi_{xx} - P^*)\, Z \Big|_{t_f} \geqslant 0.$$

COROLLARY B1.1. *If* L_1 *and* F *are quadratic functions of* x, *if* f_1 *and* L_2 *are linear in* x, *if* f_u *is independent of* x, *and if* ψ *is linear in* $x(t_f)$, *then condition* (i) *of Theorem* B1 *is necessary, and together with the strengthened existence condition* (ii) *of Theorem* B1, *is sufficient for* $V[u(\cdot)]$ *to have a strong minimum at* $\bar{u}(\cdot)$.

COROLLARY B1.2 [2]. *A sufficient condition for* $\delta^2 V^*[\delta u(\cdot)] \geqslant 0$ *is that*

[3] If A_1 is singular then differently partitioned ψ_x and $\delta x(t_f)$ must be used.

$\exists \forall t \in [t_0, t_f]$ a real, symmetric, continuously differentiable matrix function of time $P^*(\cdot)$ such that

$$H_{ux} + f_u{}^T P^* = 0 \qquad \forall t \in [t_0, t_f], \tag{116}$$

$$\dot{P}^* + P^* f_x + f_x{}^T P^* + H_{xx} = M'(t) \geqslant 0 \qquad \forall t \in [t_0, t_f], \tag{117}$$

$$Z^T(F_{xx} + \nu^T \psi_{xx} - P^*) Z \Big|_{t_f} \geqslant 0. \tag{118}$$

3. *Useful Lemmas*

LEMMA 10. *The second variation* (104)–(106) *is expressible in the equivalent canonical form*

$$\delta^2 V^*[\delta u(\cdot)] = \delta^2 \hat{V}^*[\delta u(\cdot)] \triangleq \int_{t_0}^{t_f} \delta u^T C \delta y \, dt \tag{119}$$

subject to

$$\delta \dot{y} = B \delta u, \qquad \delta y(t_0) = 0, \tag{120}$$

and

$$\psi_x \delta y \Big|_{t_f} = 0, \tag{121}$$

where C and B are given by (42), (43).

Proof. See Lemma 1. In addition we have (121) which follows from the fact that $\Phi(t_f) = I$, see (44).

LEMMA 11. *Condition* (i) *of Theorem* B1 *is equivalent to the existence $\forall t \in (t_0, t_f)$ of a real symmetric monotone increasing matrix function of time $\hat{P}^*(\cdot)$ such that*

$$C + B^T \hat{P}^* = 0 \qquad \forall t \in (t_0, t_f), \tag{122}$$

$$- Z^T \hat{P}^*(t_f^-) Z \geqslant 0. \tag{123}$$

Proof. See Lemma 2.

4. *A Related Nonsingular Second Variation*

Consider the nonsingular variational problem

$$\delta^2 V_N^*[\delta u(\cdot), \epsilon] = \delta^2 \hat{V}^*[\delta u(\cdot)] + \frac{1}{2\epsilon} \int_{t_0}^{t_f} \delta u^T \delta u \, dt \tag{124}$$

$$= \int_{t_0}^{t_f} \left\{ \delta u^T C \delta y + \frac{1}{2\epsilon} \delta u^T \delta u \right\} dt \tag{125}$$

subject to

$$\delta \dot{y} = B \delta u, \qquad \delta y(t_0) = 0, \tag{126}$$

and the linearized terminal constraint

$$\psi_x \delta y \Big|_{t_f} = 0. \tag{127}$$

THEOREM 6 (Gelfand and Fomin, Ref. [11, p. 123], Breakwell and Ho, Ref. [15]). *If the quadratic functional* $\delta^2 \hat{V}^*[\delta u(\cdot)]$ *is nonnegative then the matrix Riccati differential equation*

$$- \dot{S}_\epsilon = - (C + B^T S_\epsilon)^T (C + B^T S_\epsilon) \epsilon \tag{128}$$

associated with $\delta^2 V_N{}^*[\delta u(\cdot), \epsilon]$ *has a solution* $S_\epsilon(t)$ *which exists* $\forall t \in [t_0, t_f)$. *In a neighborhood of* t_f, S_ϵ *is given by* [17]

$$S_\epsilon(t) = W_\epsilon(t) - N_\epsilon(t) M_\epsilon^{-1}(t) N_\epsilon{}^T(t), \tag{129}$$

where

$$- \dot{W}_\epsilon = - (C + B^T W_\epsilon)^T (C + B^T W_\epsilon) \epsilon, \qquad W(t_f) = 0, \tag{130}$$

$$\dot{M}_\epsilon = N_\epsilon{}^T B B^T N_\epsilon \epsilon, \qquad M_\epsilon(t_f) = 0, \tag{131}$$

and

$$\dot{N}_\epsilon = (C + B^T W_\epsilon)^T B^T N_\epsilon \epsilon, \qquad N_\epsilon(t_f) = \psi_x{}^T. \tag{132}$$

Note that $M_\epsilon(t)$, $t < t_f$ is invertible by Assumption 3.

THEOREM 7. *If the matrix Riccati differential equation* (128) *has a solution* $S_\epsilon(\tau)$ *which exists for all* $\tau \in [t_0, t_f)$ *then*

$$\begin{aligned} \min_{\delta u(\cdot)} \delta^2 V_N^{*\tau}[\delta y(\tau), \delta u(\cdot), \epsilon, \tau] &\triangleq \delta^2 V^{*0}[\delta y(\tau), \epsilon, \tau] \\ &= \tfrac{1}{2} \delta y^T(\tau) S_\epsilon(\tau) \delta y(\tau). \end{aligned} \tag{133}$$

Proof. See Ref. [17, pp. 183–184].

LEMMA 12. $S_\infty(t) \triangleq \lim_{\epsilon \to \infty} S_\epsilon(t)$ *exists* $\forall t \in (t_0, t_f)$ *and is a real symmetric monotone increasing matrix function of time such that*

$$C + B^T S_\infty = 0 \qquad \text{a.e. } in\ [t_0, t_f]. \tag{134}$$

Proof. The proof is very similar to the proof of Lemmas 4–8 and so is not described here.

LEMMA 13.

$$\lim_{\epsilon\to\infty}\lim_{t\to t_f} Z^T S_\epsilon(t)\, Z = Z^T S_\infty(t_f)\, Z = 0, \tag{135}$$

where Z is defined in (110).

Proof. We have

$$S_\epsilon(t) = W_\epsilon(t) - N_\epsilon(t)\, M_\epsilon^{-1}(t)\, N_\epsilon^T(t). \tag{136}$$

Since N_ϵ obeys a linear homogeneous differential equation we have

$$S_\epsilon(t) = W_\epsilon(t) - \psi_x{}^T\theta_\epsilon{}^T(t)\, M_\epsilon^{-1}(t)\, \theta_\epsilon(t)\, \psi_x\,, \tag{137}$$

where

$$\dot{\theta}_\epsilon = (C + B^T W_\epsilon)^T B^T \theta_\epsilon \epsilon, \qquad \theta_\epsilon(t_f) = I. \tag{138}$$

From (110), (137) it is clear that

$$Z^T S_\epsilon(t)\, Z = Z^T W_\epsilon(t)\, Z \tag{139}$$

because

$$\psi_x Z \equiv 0. \tag{140}$$

Thus, from (139),

$$Z^T S_\infty(t_f)\, Z = Z^T W_\infty(t_f)\, Z = 0 \tag{141}$$

since

$$W_\epsilon(t_f) = 0, \forall\epsilon. \tag{142}$$

LEMMA 14.

$$Z^T S_\infty(t_f^-)\, Z \leqslant 0. \tag{143}$$

Proof. From Lemma 12 $S_\infty(t)$ is monotone increasing. Lemma 14 then follows from Lemma 13.

5. *Proof of Theorem* B1.

(i) *Necessary Conditions.* If Assumptions 1–7 are satisfied and if $\delta^2\tilde{V}[\delta u(\cdot)] \geqslant 0$ for all admissible $\delta u(\cdot)$ then by Lemmas 12–14 $\exists\forall t \in (t_0, t_f)$ a real symmetric monotone increasing matrix function of time such that

$$C + B^T S_\infty = 0 \qquad \text{a.e. in } [t_0, t_f], \tag{144}$$

$$Z^T S_\infty(t_f^-)\, Z \leqslant 0. \tag{145}$$

Defining

$$\hat{P}^*(t) = S_\infty(t), \qquad t \in (t_0, t_f), \tag{146}$$

yields

$$\hat{P}^*(t_f^-) = S_\infty(t_f^-) \tag{147}$$

and

$$C + B^T\hat{P}^* = 0 \qquad \text{a.e. in } [t_0, t_f], \tag{148}$$

$$Z^T\hat{P}^*(t_f^-)\, Z \leqslant 0. \tag{149}$$

Using the same argument on (148) as was used in Section II.5 on equation (93) shows that (148), (149) imply (122), (123) which by Lemma 11 imply the necessary conditions of Theorem B1.

(ii) *Sufficient Condition.* Suppose that (122) is satisfied by a $\hat{P}^*(t)$ which exists $\forall t \in [t_0, t_f]$ (strengthened existence condition) such that $-Z^T\hat{P}^*Z\,|_{t_f} \geqslant 0$.

Adjoin the linearized dynamics to the canonical second variation as follows:

$$\delta^2\hat{V}^*[\delta u(\cdot)] = \delta^2\tilde{V}^*[\delta u(\cdot)] \triangleq \int_{t_0}^{t_f} \{\delta u^T C\delta y + \delta y^T\hat{P}^*(B\delta u - \delta\dot{y})\}\, dt \tag{150}$$

$$= -\int_{t_0}^{t_f} \delta y^T\hat{P}^*\delta\dot{y}\, dt \tag{151}$$

because of (122). Integrating (151) by parts yields

$$\delta^2\hat{V}^*[\delta u(\cdot)] = \int_{t_0}^{t_f} \tfrac{1}{2}\,\delta y^T\, d\hat{P}^*(t)\,\delta y - \tfrac{1}{2}\,\delta y^T\hat{P}^*\delta y\,\Big|_{t_f} \tag{152}$$

where the integral is in the Stieltjies sense.

Using (110) and the fact that $\delta x(t_f) = \delta y(t_f)$ yields

$$\delta^2\hat{V}^*[\delta u(\cdot)] = \int_{t_0}^{t_f} \tfrac{1}{2}\,\delta y^T\, d\hat{P}^*(t)\,\delta y - \tfrac{1}{2}\,[\delta y^{n-s}]^T\, Z^T\hat{P}^*Z[\delta y^{n-s}]\,\Big|_{t_f}. \tag{153}$$

Since $\hat{P}^*(t)$ is monotone increasing the integral is nonnegative and since $-Z^T\hat{P}^*Z\,|_{t_f} \geqslant 0$ the boundary term is nonnegative. This concludes the sufficiency proof.

Note. This sufficiency condition for the constrained terminal state problem may appear to be stringent because a $\hat{P}^*(t_f)$ is required to exist, but

$S_\infty(t_f)$ is undefined. However, as in the free terminal state case it is the authors' opinion that the gap between the necessary and the sufficient condition is minimal. This is supported by the following (slightly altered) sufficiency theorem due to Brockett, Ref. [18, p. 140] (a proof of necessity of this theorem is not given in Ref. [18] but appears to be straightforward).

THEOREM 7. (cf. Theorem 6). *If there exists a symmetric boundary condition* $S_\epsilon(t_f)$ *such that the solution of* (128) *exists* $\forall t \in [t_0, t_f]$, *then* $\delta^2 V_N^*[\delta u(\cdot), \epsilon]$ *is positive definite.*[4]

Of course the solution of (128) with finite boundary condition is not the same as the weighting matrix (136) of the quadratic optimal value function. This is consistent with the fact that $\hat{P}^*(t_f)$ in our sufficiency condition cannot be identified with $S_\infty(t_f)$; that is, in our sufficiency condition for the fixed terminal state problem we lose the identification of $\hat{P}^*(\cdot)$ being the limit of the weighting matrix of the quadratic optimal value function as $\epsilon \to \infty$.

IV. Relation to Existing Necessary Conditions of Optimality

Known necessary conditions for singular problems can be deduced from our theorems. Here we give the most important ones.

THEOREM C1 [Robbins Ref. [7], Goh, Ref. [10]). *Under Assumptions* 1–5 *a necessary condition for* $V[u(\cdot)]$ *to have a minimum at* $\bar{u}(\cdot)$ *is that* $H_{ux} f_u$ *be symmetric for all* $t \in [t_0, t_f]$.

Proof. From (32)

$$H_{ux} f_u + f_u^T P f_u = 0 \qquad \forall t \in (t_0, t_f]. \tag{154}$$

Since P is symmetric it follows that $H_{ux} f_u$ is symmetric in $(t_0, t_f]$. Assumption 2 implies that indeed $H_{ux} f_u$ is symmetric in $[t_0, t_f]$.

THEOREM C2 (Jacobson, Ref. [1]). *Under Assumptions* 1–5 *a necessary condition for* $V[u(\cdot)]$ *to have a minimum (unconstrained terminal state case) at* $\bar{u}(\cdot)$ *is that*

$$(H_{ux} + f_u^T Q) f_u \geqslant 0 \qquad \forall t \in [t_0, t_f], \tag{155}$$

where Q *satisfies* (36).

Proof. From (32) and (34)

$$H_{ux} f_u + f_u^T Q f_u + f_u^T \Phi^T \hat{P} \Phi f_u = 0 \qquad \forall t \in (t_0, t_f]. \tag{156}$$

[4] Brockett treats the case where $\psi_x = I$.

Since $\hat{P}(t) \leqslant 0 \; \forall t \in [t_0, t_f]$ it follows from (156) that

$$(H_{ux} + f_u{}^T Q) f_u \geqslant 0 \qquad \forall t \in (t_0, t_f]. \tag{157}$$

Since H_{ux}, f_u, Q are continuous in time (155) follows.

THEOREM C3 (Kelley, Ref. [6], Robbins, Ref. [7], generalized Legendre–Clebsch condition). *Under Assumptions* 1–5 *a necessary condition for* $V[u(\cdot)]$ *to have a minimum at* $\bar{u}(\cdot)$ *is that*

$$(-1) \frac{\partial}{\partial u} \ddot{H}_u \geqslant 0 \qquad \forall t \in [t_0, t_f]. \tag{158}$$

Proof. From (32), and post-multiplying by f_u, we obtain[5]

$$[(\dot{H}_{ux} + \dot{f}_u{}^T P) f_u + (H_{ux} + f_u{}^T P) \dot{f}_u]\, dt + f_u{}^T\, dP f_u = 0 \tag{159}$$

where, from (34)–(36),

$$dP = (-H_{xx} - f_x{}^T P - P f_x)\, dt + \Phi^T\, d\hat{P} \Phi. \tag{160}$$

Using (32) and (160) in (159) yields

$$\begin{aligned}(\dot{H}_{ux} f_u + H_{ux} \dot{f}_u - \dot{f}_u{}^T H_{xu} - H_{ux} \dot{f}_u - f_u{}^T H_{xx} f_u + f_u{}^T f_x{}^T H_{xu} + H_{ux} f_x f_u)\, dt \\ = - f_u{}^T \Phi^T\, d\hat{P} \Phi f_u\,. \end{aligned} \tag{161}$$

The left side of (161) is just $[(\partial/\partial u)\, \ddot{H}_u]\, dt$. Since $\hat{P}$ is monotone increasing in t we obtain from (161)

$$(-1) \frac{\partial}{\partial u} \ddot{H}_u \geqslant 0 \qquad \forall t \in (t_0, t_f). \tag{162}$$

By the assumed continuity of $(\partial/\partial u)\, \ddot{H}_u$ (162) implies

$$(-1) \frac{\partial}{\partial u} \ddot{H}_u \geqslant 0 \qquad \forall t \in [t_0, t_f]. \tag{163}$$

Note. If H_{xx}, f_x, f_u, H_{ux} are assumed smooth in t then the general form of the generalized Legendre–Clebsch condition can be deduced in the manner outlined above,

$$(-1)^p \frac{\partial}{\partial u} \left[\frac{d^{2p}}{dt^{2p}} H_u \right] \geqslant 0. \tag{164}$$

[5] Here dP is the increment in P in time dt.

V. Relation to the Generalized Jacobi (Riccati Equation) Necessary Condition

In Ref. [5] Kelley's transformation technique is used to transform the singular second variation into a nonsingular one in a reduced state space. A condition for this to succeed is that (163) be satisfied with strict inequality. It is then shown that the Riccati differential equation associated with this nonsingular problem implies the conditions of Corollary A1.2, i.e., these conditions are necessary as well as sufficient for a large class of problems. (It should be possible to prove this via our limit approach; one would only have to show that $S_\infty(t)$ is continuously differentiable with respect to t, $t \in (t_0, t_f)$.)

In Ref. [5] sufficient conditions are given to ensure that Theorem A2 holds. We state one set of these conditions here:

Theorem D1. *If the conditions of Corollary* A1.2 *are satisfied then the following are sufficient to ensure that*

$$\delta^2 V[\delta u(\cdot)] \geqslant 0 \Rightarrow V[\bar{u}(\cdot) + \delta u(\cdot)] \geqslant V[\bar{u}(\cdot)]: \tag{165}$$

(a) *Strengthened generalized Legendre-Clebsch condition,*

$$-\frac{\partial}{\partial u} \ddot{H}_u > 0 \qquad \forall t \in [t_0, t_f]; \tag{166}$$

(b) *Strengthened Jacobson condition at the terminal time,*

$$(H_{ux} + f_u^T F_{xx}) f_u \Big|_{t_f} > 0. \tag{167}$$

In fact (166), (167) together with the conditions of Corollary A1.2 are sufficient to ensure that the *transformed second variation* is strongly positive with respect to the control variable in the transformed space.

VI. Conclusion

Necessary and sufficient conditions for optimality for singular control problems are obtained by studying the limiting behavior of a nonsingular second variation. This nonsingular second variation is constructed in such a way that it tends to the singular second variation as a parameter approaches infinity. Optimality conditions for both unconstrained and constrained terminal state problems are obtained.

The optimality conditions derived in this paper are very similar to certain sufficient conditions of Jacobson [2].[6] In a companion paper [5] it is shown that Jacobson's sufficient conditions are also necessary for a large class of singular optimal control problems; moreover, satisfaction of these conditions is shown to be equivalent to the existence of a solution of a certain matrix Riccati differential equation.

The closing sections of the present paper relate the necessary and sufficient conditions to known necessary conditions. In particular the important necessary conditions of Robbins [7], Goh [8], Kelley et al. [6], and Jacobson [1] follow easily from these new results.

Acknowledgments

D. H. Jacobson wishes to thank D. G. M. Anderson for useful discussions.

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